*International Journal of Theoretical Physics, Vol. 38, No. 4, 1999*

# Black Hole Radiation and *S*-matrix

**J. G. Russo**[1]

*Received September 15, 1998*

The existence of an $S$-matrix below the threshold of black hole formation would be enough to exhibit, through its analytic structure, eventual thresholds for the creation of new objects and to describe, through analytic continuation, the physics above them in a unitary framework. In the context of a two-dimensional exactly soluble model, the semiclassical dynamics of quantum black holes is obtained by analytically continuing the description of the regime where no black hole is formed. The resulting spectrum of outgoing radiation departs from the one predicted by the Hawking model by the time the outgoing modes arise from the horizon with Planck-order frequencies. The theory predicts an unconventional scenario for the evolution: black holes only radiate out an energy of Planck mass order, stabilizing after a transitory period. A similar picture is obtained in $3 + 1$ dimensions with spherical symmetry.

## 1. INTRODUCTION

We shall revisit a problem that has been discussed to some extent in the recent years, namely whether the process of black hole formation and evaporation [1] can be described by a unitary $S$-matrix [2–6; see ref. 3 for a review]. The present work is based on refs. 7 and 8.

Black hole physics provides a setting for the study of the interplay between general relativity and quantum mechanics. In particular, it appears difficult to reconcile the apparently thermal evaporation of a black hole formed in gravitational collapse with the Hamiltonian evolution of pure quantum mechanical states [9]. In the Hawking model [1], after a black hole has completely evaporated, the outgoing matter is constituted by thermal radiation and all the quantum mechanical information about the original collapsing matter is lost. It has been argued that, due to backreaction effects, the Hawking model may break down long before the evaporation is complete [2, 3, 6, 10–13]. Because of an exponential redshift, the outgoing modes arise from a reservoir of trans-Planckian ener-

[1] Theoretical Physics Group, Imperial College, London SW7 2BZ, England.



0020-7748/99/0400-120 7$16.00/0 

gies; if a Planck-scale cutoff is imposed before the horizon, it seems that there would be only a scarce amount of outgoing modes, and black holes would lose an insignificant mass by evaporation. Lacking the fundamental short-distance theory, by the time the outgoing modes arise from the horizon with Planck frequencies, some extra assumption is needed. Extrapolating the Hawking radiation into this region leads to paradoxes, such as loss of quantum coherence. 't Hooft has proposed to start from the postulate that an $S$-matrix exists, and then find out what new kind of matter is required ("the $S$-matrix ansatz" [3]). In particular, the unitarity property of the $S$-matrix seems to require a huge reduction of matter degrees of freedom (proportional to the area, instead of the field-theoretic dependence on volume) to the extent that one space-time dimension would be superfluous [14]. However, unitarity is not the only implication of having an $S$-matrix: an $S$-matrix also requires that the physics above the thresholds is described by the same (analytically continued) formulas that govern the physics below the thresholds. Surprisingly, we will find that the assumption of analyticity suggests a black hole evolution quite different from the standard one.

Let the event horizon be located at a retarded (Kruskal) time $U = -p$. The energy that has been radiated up to a time $U_1$ is given by (see Section 5)

$$E_{\text{out}} \cong -\frac{1}{MG}\log(U_1 + p) \tag{1}$$

How close to the horizon does one need to extrapolate the Hawking model to have evaporation? From Eq. (1) we see that $E_{\text{out}} \cong M$ implies $U_1 + p \cong \exp(-GM^2)$. That is, the Hawking model needs to be extrapolated up to exponentially small proper distances from the horizon $\sim \exp(-GM^2)$. An outgoing mode with frequency $w_\infty \sim (MG)^{-1}$ thus has frequency $w_0 \sim (MG)^{-1} \exp(GM^2)$ at the moment it arises from the horizon (modulo power-like corrections). For a macroscopic black hole with $M^2 \gg 1/G$, this means $w_0 \gg M$! Corrections are first expected when $w_0 \sim 1/\sqrt{G}$ (this would be the location of the "stretched horizon" in ref. 11; the different horizons that are relevant to the evaporation problem are discussed in ref. 15). By that time, almost *nothing* has been radiated, $E_{\text{out}} = O(1/MG) < m_{\text{planck}}$. According to ref. 3, quantum gravity effects are such that there is radiation with $w_0 > 1/\sqrt{G}$, but it is no longer thermal. It should contain all the original quantum mechanical information encoded in subtle correlations, so that unitarity in the evolution is preserved. This requires the notion of complementarity, as emphasized in ref. 11; indeed, an inertial observer falling into a large black hole does not see any strong quantum gravity effects in crossing the horizon (since curvatures are much smaller than $G^{-1}$). It is clear that the description of physics according to the out and infalling observers must be very different; in particular, the infalling observer sees no Hawking radiation at all. A description based on an $S$-matrix approach can only be appropriate for the out observer.

The investigation of the problem in the context of 3 + 1 Einstein gravity faces two inconvenient issues. Einstein theory is not renormalizable and including backreaction effects is very complicated. A simplified framework is obtained by restricting the dynamics to spherically symmetric configurations, i.e., metrics of the form

$$ds^2 = g_{ij}(x^0, x^1)\, dx^i\, dx^j + r^2(x^0, x^1)\, d\Omega^{D-2}, \qquad i, j = 0, 1 \tag{2}$$

This leads to a two-dimensional theory containing a 2-tensor $g_{ij}$ and a scalar $\phi$, $r^2(x) = e^{-2\phi(x)}$. The most general local action containing terms with no more than two derivatives is given by

$$S = \int d^2x \sqrt{-g}e^{-2\phi}[R + F(\phi)(\partial\phi)^2 + V(\phi)] \tag{3}$$

The action (3) includes as special cases the dimensional reduction of 3 + 1 Einstein–Hilbert action,

$$S_{\text{einstein}} = \int d^2x \sqrt{-g}e^{-2\phi}[R + 2(\partial\phi)^2 + 2e^{2\phi}] \tag{4}$$

and the low-energy effective action of string theory,

$$S_{\text{string}} = \int d^2x \sqrt{-g}e^{-2\phi}[R + 4(\partial\phi)^2 + 4\lambda^2] \tag{5}$$

The model (5) has black hole solutions, where the metric and dilaton are given by ($\lambda = 1$)

$$\begin{aligned} ds^2 &= \frac{dx^+\, dx^-}{M - x^+x^-} = \frac{1}{1 - Me^{-2\sigma}}(-d\tau^2 + d\sigma^2) \\ &= -(1 - Me^{-2x})\, dt^2 + (1 - Me^{-2x})^{-1}\, dx^2, \qquad \phi = -x \end{aligned} \tag{6}$$

In two-dimensional gravity the gravitational coupling $e^{2\phi}$ asymptotically decreases like $1/r^2$. This accounts, in particular, for the $1/r^2$ dilution effect of the energy density of an in/outgoing spherical shell. Two-dimensional dilaton gravity embodies all the features of the Hawking model of gravitational collapse. Nevertheless, angular coordinates might play an important role in the dynamics of the evaporation of (3 + 1)-dimensional black holes and they are ignored in the reduction (see refs. 3 and 14).

## 2. EXACTLY SOLVABLE MODEL

By including conformal matter and the one-loop anomaly term $R(\nabla^2)^{-1} R$ in the action (5), in ref. 16 a model for the semiclassical description of the backreaction of the Hawking radiation on the geometry was constructed. The equations of motion of the resulting model [16] cannot be solved in a closed form, but there is a similar model which is exactly solvable [17],

$$S = \frac{1}{2\pi}\int d^2x\,\sqrt{-g}\left[e^{-2\phi}(R + 4(\nabla\phi)^2 + 4\lambda^2) - \frac{1}{2}\sum_{i=0}^{N}(\nabla f_i)^2 - \kappa(2\phi R + R(\nabla^2)^{-1}R)\right], \qquad \kappa = \frac{N}{48} \tag{7}$$

By fixing the conformal gauge $g_{\pm\pm} = 0$, $g_{+-} = -\frac{1}{2}e^{2\rho}$, and making a field redefinition

$$\chi = 4\kappa\rho + e^{-2\phi} - 2\kappa\phi, \qquad \Omega = e^{-2\phi} + 2\kappa\phi \tag{8}$$

we find that the action (7) takes the simple form

$$S\int d^2x\left[\frac{1}{4\kappa}(\partial_+\Omega\partial_-\Omega - \partial_+\chi\partial_-\chi) + \lambda^2 e^{(1/2\kappa)(\chi-\Omega)} + \frac{1}{2}\sum_{i=0}^{N}\partial_+ f_i\partial_- f_i\right] \tag{9}$$

In addition, we have the constraints originating from the $g_{\pm\pm}$ equations of motion:

$$t_\pm(x^\pm) = \frac{1}{4\kappa}(-\partial_\pm\chi\partial_\pm\chi + \partial_\pm\Omega\partial_\pm\Omega) + \partial_\pm^2\chi + \frac{1}{2}\sum_{i=0}^{N}\partial_\pm f_i\partial_\pm f_i \tag{10}$$

The functions $t_\pm(x^\pm)$ are determined by boundary conditions.

In the "Kruskal" gauge $\chi = \Omega$, it is easy to find the general solution to the equations of motion of (9) with the constraints (10) (representing a general distribution of collapsing matter). It is given by

$$\begin{aligned}
\Omega = \chi &= -\lambda^2 x^+(x^- + \lambda^{-2}P_+(x^+)) - \kappa\log(-\lambda^2 x^+ x^-) + \lambda^{-1}M(x^+)\\
P_+(x^+) &= \int_0^{x^+} dx^+\, T_{++}(x^+)\\
M(x^+) &= \lambda\int_0^{x^+} dx^+\, x^+ T_{++}(x^+)\\
T_{++}(x^+) &= \frac{1}{2}\sum_{i=0}^{N}\partial_+ f_i\partial_+ f_i
\end{aligned} \tag{11}$$

The vacuum solution is obtained by setting $T_{++} = 0$,

$$e^{-2\phi} = e^{-2\rho} = -\lambda^2 x^+ x^- \tag{12}$$

or, introducing Minkowskian coordinates $\sigma^{\pm}$ by $\lambda x^{\pm} = \pm e^{\pm\lambda\sigma^{\pm}}$, one gets $\rho = 0$, $\phi = -\lambda\sigma$, and $\sigma^{\pm} = \tau \pm \sigma$. The scalar curvature of the general geometry can be written as [see Eq. (8)]

$$R = 8e^{-2\rho}\partial_+\partial_-\rho = 8e^{-2\rho}\frac{1}{\Omega'(\phi)}(\partial_+\partial_-\chi - 4\partial_+\phi\partial_-\phi e^{-2\phi}) \tag{13}$$

In this form we see that, generically, there will be a curvature singularity at $\phi = \phi_{cr} = -\frac{1}{2}\log\kappa$, where $\Omega'(\phi) = 0$. Inspection of Eq. (11) shows that there are two possible regimes:

$$\text{subcritical regime:} \qquad T_{++}(x^+) < T^{cr}_{++}(x^+),$$

$$T^{cr}_{++}(x^+) = \frac{\kappa}{x^{+2}}$$

$$\text{supercritical regime:} \qquad T_{++}(x^+) > T^{cr}_{++}(x^+)$$

In the subcritical regime the line $\phi = \phi_{cr}$ is timelike. In the supercritical regime it becomes spacelike, and the causal structure of the geometry is given by the usual Carter–Penrose diagram of the Hawking model of gravitational collapse [17].

Let us assume that originally the geometry is the linear dilaton vacuum; at some time $x_0^+$, an incoming energy flux $T_{++}(x^+) < T^{cr}_{++}(x^+)$ is turned on and at some later time $x_1^+$ it is turned off. There are three different regions, as indicated in Fig. 1. In region (i), the solution is given by Eq. (11),

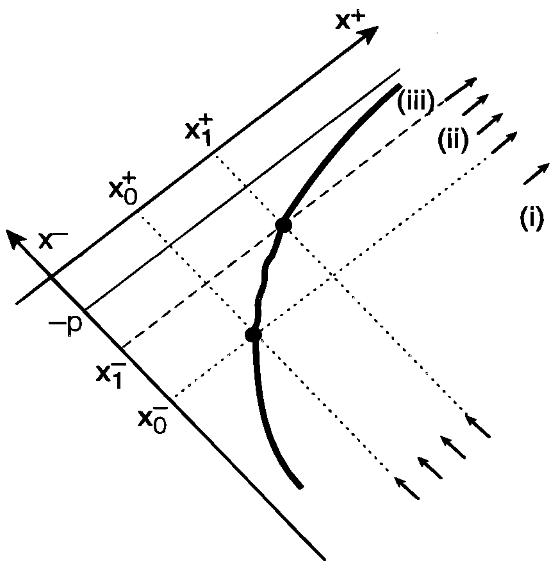


**Fig. 1.** Kruskal diagram in the subcritical regime.

which is completely specified by the asymptotic boundary conditions and by demanding a continuous matching with the linear dilaton vacuum in the infalling line $x^+ = x_0^+$. In region (ii) the boundary $\phi = \phi_{cr}$ is timelike and boundary conditions are needed in order to determine the evolution. Continuity along the line $x^- = x_0^-$ requires that the solution in region (ii) be of the form

$$\Omega^{(ii)}(x^+, x^-) = \Omega^{(i)}(x^+, x^-) + F(x^-)$$

with $F(x_0^-) = 0$. The "reflecting" boundary conditions of ref. 17 follow from the requirement that the curvature is finite at the boundary line. Using Eqs. (8) and (13), one finds the conditions

$$\partial_+ \Omega|_{\phi=\phi_{cr}} = \partial_- \Omega|_{\phi=\phi_{cr}} = 0 \tag{14}$$

As a result, the function $F(x^-)$ is determined to be

$$F(x^-) = \kappa \ln \left[ \frac{x^-}{x^- + \lambda^{-2} P_+(\hat{x}^+)} \right] - \lambda^{-1} M(\hat{x}^+) \tag{15}$$

where $\hat{x}^+ = \hat{x}^+ (x^-)$ describes the boundary curve:

$$-\lambda^2 x^+ (x^- + \lambda^{-2} P_+(\hat{x}^+)) = \kappa \tag{16}$$

Finally, in region (iii) the geometry is matched with the vacuum:

$$\begin{aligned} \Omega^{(iii)} &= \chi^{(iii)} \\ &= -\lambda^2 x^+(x^- + p) - \kappa \ln[-\lambda^2 x^+(x^- + p)], \\ p &\equiv \lambda^{-2} P_+(x_1^+) \end{aligned}$$

In Minkowski coordinates, the geometry in (iii) is simply given by

$$ds^2 = -d\sigma^+ \, d\sigma^-, \qquad \phi = -\lambda\sigma, \qquad \lambda(x^- + p) = -e^{\lambda\sigma^-}, \qquad \lambda x^+ = e^{\lambda\sigma^+}$$

Let us now determine the outgoing energy density fluxes. Using the constraints (10), we find

$$T_{--}^{(i)}(x^-) = \kappa \left[ \frac{1}{(x^- + p)^2} - \frac{1}{x^{-2}} \right] \tag{17}$$

$$T_{--}^{(ii)}(x^-) = \kappa \frac{1}{(x^- + p)^2} - \frac{\lambda^4}{\kappa/(\hat{x}^+)^2 - T_{++}(\hat{x}^+)} \tag{18}$$

$$T^{(iii)}(x^-) = 0 \tag{19}$$

## 3. BLACK HOLES BY ANALYTIC CONTINUATION

As discussed in Section 1, by the time backreaction effects of the Hawking radiation are expected to be important, some additional input is needed in order to determine the subsequent evolution of the geometry. The assumption here is based on analytically continuing the formulas for the subcritical regime obtained in the preceding section [7].

### 3.1. Outgoing Energy-Density Flux

In the subcritical theory, from Eqs. (17) and (18) one obtains the following expressions for the total energies radiated in regions (i) and (ii):

$$E_{\text{out}}^{(\text{i})} = -\lambda \int_{-\infty}^{x_0^-} dx^- \, (x^- + p) T_{--}^{(\text{i})} = -\frac{\kappa\lambda p}{x_0^-} - \kappa\lambda \ln\left(1 + \frac{p}{x_0^-}\right) \quad (20)$$

$$E_{\text{out}}^{(\text{ii})} = -\lambda \int_{x_0^-}^{x_1^-} dx^- \, (x^- + p) T_{--}^{(\text{ii})} = m + \frac{\kappa\lambda p}{x_0^-} + \kappa\lambda \ln\left(1 + \frac{p}{x_0^-}\right) \quad (21)$$

Consider the intermediate regime (Fig. 2), where there is black hole formation and yet $p < |x_0^-|$, so that we can continue the above expressions in a straightforward way (this intermediate regime physically corresponds to the formation of Planck-size black holes). Let us split the integral (20) as $E_{\text{out}}^{(\text{i})} = E_{\text{out}}^{(\text{a})} + E_{\text{out}}^{(1\text{b})}$, where

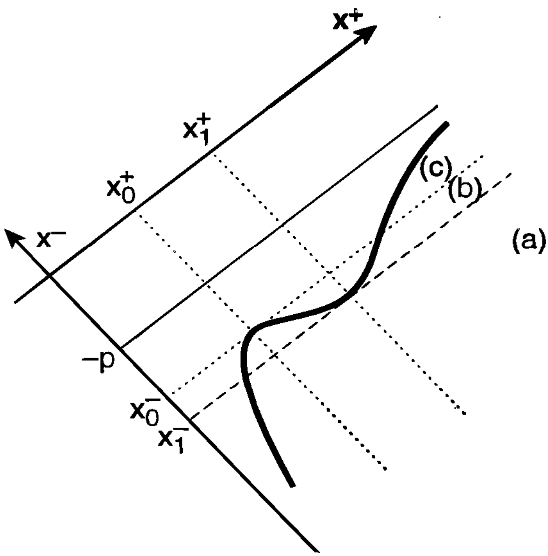


**Fig. 2.** Intermediate regime. The thick line represents the apparent horizon given by Eq. (16)

$$E_{\rm out}^{(a)} = -\lambda \int_{-\infty}^{x_1^-} dx^- \,(x^- + p) T_{--}^{(i)} = -\frac{\kappa\lambda p}{x_1^-} - \kappa\lambda \ln\left(1 + \frac{p}{x_1^-}\right) \tag{22}$$

$$E_{\rm out}^{(1b)} = -\lambda \int_{x_1^-}^{x_0^-} dx^- \,(x^- + p) T_{--}^{(i)} = -\frac{\kappa\lambda p}{x_0^-} + \frac{\kappa\lambda p}{x_1^-} - \kappa\lambda \ln\frac{(1 + p/x_0^-)}{(1 + p/x_1^-)} \tag{23}$$

The first integral gives the energy radiated in region (a) of Fig. 2. The second integral contributes to the radiation in region (b).

The total energy radiated in region (b) is obtained by adding $E_{\rm out}^{(ii)}$. Since now $x_1^- < x_0^-$, it is convenient to write this integral in the following way:

$$E_{\rm out}^{(ii)} = -\lambda \int_{x_1^-}^{x_0^-} dx^- \,(x^- + p)\,(-\,T_{--}^{(ii)}) = E_{\rm out}^{(2b)}$$

$$= m + \frac{\lambda\kappa p}{x_0^-} + \kappa\lambda \ln\left(1 + \frac{p}{x_0^-}\right) \tag{24}$$

Thus

$$E_{\rm out}^{(b)} = E_{\rm out}^{(1b)} + E_{\rm out}^{(2b)} = -\lambda \int_{x_1^-}^{x_0^-} dx^- \,(x^- + p)\, \tilde{T}_{--}^{(b)} \tag{25}$$

with

$$\tilde{T}_{--}^{(b)} \equiv T_{--}^{(i)} - T_{--}^{(ii)} \tag{26}$$

Hence

$$E_{\rm out}^{(b)} = m + \frac{\lambda\kappa p}{x_1^-} + \kappa\lambda \ln\left(1 + \frac{p}{x_1^-}\right) \tag{27}$$

Clearly, $E_{\rm out}^{(a)} + E_{\rm out}^{(b)} = m$, so that the whole incoming energy has been radiated. This means that these "small" black holes evaporate completely.

It should be noticed that in region (b) (i.e., in the region in causal contact with the apparent horizon) the $T_{--}$ arising in the model of ref. 17 (which is a 1 + 1-dimensional counterpart of the Hawking model) does not coincide with the continuation of the subcritical formulas given by Eq. (25). Indeed, in ref. 17 the $T_{--}$ keeps being $T_{--}^{(i)}$ until the geometry is matched with the vacuum. Although in both cases the original energy is completely radiated, the structure of the outgoing energy-density fluxes of the two models is different already for this intermediate regime, where the continuation is straightforward.

### 3.2. Apparent Horizon as a Boundary

In the subcritical regime the boundary conditions (14) or, equivalently,

$$\Omega = \Omega_{cr}, \qquad \partial_+ \Omega = 0$$

can be implemented simultaneously on some line. Above the threshold, the curve defined by $\Omega = \Omega_{cr}$ splits from the curve defined by $\partial_+\Omega = 0$ (which is the apparent horizon). Thus there seem to be two possible supercritical theories, according to where the boundary is placed:

(A) $\Omega = \Omega_{cr}$. This choice leads to the model of ref. 17, which, as pointed out above, is not the analytic continuation of the subcritical theory.

(B) $\partial_+\Omega = 0$. We will now show that this choice reproduces the results that were previously obtained by a simple continuation of the subcritical formulas. This would thus be the natural choice for an *S*-matrix description.

So let the boundary be at the apparent horizon $\partial_+\Omega = 0$. As before, we consider the situation where the incoming supercritical energy-density flux $T_{++}(x^+)$ is turned on at $x_0^+$, and it is turned off at a later time $x_1^+$. The boundary $\partial_+\Omega = 0$ becomes timelike for $x^+ > x_1^+$, and boundary conditions are needed in order to determine the evolution of the geometry in region (b) (see Figs. 2 and 3). Continuity along the line $x^- = x_1^-$ requires that

$$\Omega^{(b)}(x^+, x^-) = \Omega^{(a)}(x^+, x^-) + F(x^-), \qquad F(x_0^-) = 0 \tag{28}$$

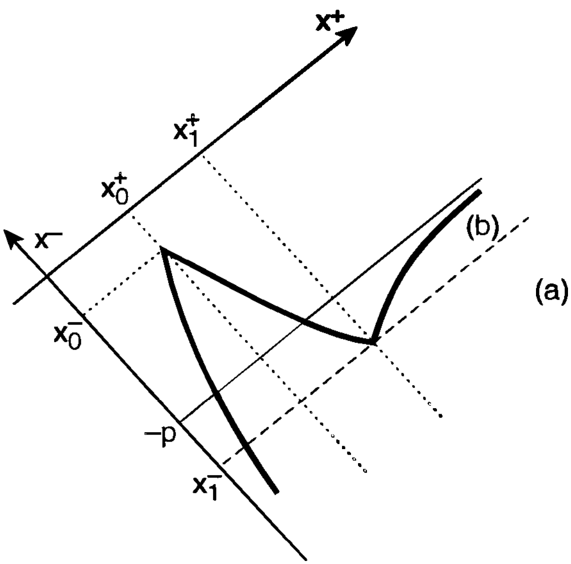


**Fig. 3.** Apparent horizon in the supercritical regime.

We need to generalize the expression (15) for the case when there is some energy stored in the geometry by the time the boundary becomes timelike, that is,

$$F(x^-) = \kappa \ln \left( \frac{x^- + \lambda^{-2} P_+(u)}{x^- + p} \right) + \lambda^{-1} M(u) - \lambda^{-1} M(x_1^+) \qquad (29)$$

with $u(x^-)$ given by the branch $x_0^+ < u < x_1^+$ of the solution to the equation

$$-\lambda^2 u(x^- + \lambda^{-2} P_+(u)) = \kappa$$

The matching between regions (a) and (b) is smoother as compared to the model of ref. 17; there is no outgoing shockwave (namely no "thunderpop" [17]):

$$T_{--}(x_1^- + \epsilon) - T_{--}(x_1^- - \epsilon) = -\frac{dF}{dx^-}\,\delta(x - x_1^-) = 0$$

since $dF/dx^-$ vanishes at $x^- = x_1^-$.

Let us now determine the outgoing energy-density fluxes. Consider first the case of "small" black holes, i.e., the intermediate regime with $p < |x_0^-|$ (Fig. 2). In region (c) the geometry is the linear dilaton vacuum,

$$\Omega^{(c)}(x^+, x^-) = -\lambda^2 x^+(x^- + p) + \kappa \ln(-\lambda^2 x^+(x^- + p)) \qquad (30)$$

Using Eq. (10), we find the following expressions for the energy-momentum tensor in the different regions:

$$T_{--}^{(a)}(x^-) = \kappa \left[ \frac{1}{(x^- + p)^2} - \frac{1}{x^{-2}} \right] \qquad (31)$$

$$T_{--}^{(b)}(x^-) = \lambda^2 \frac{du}{dx^-} - \frac{\kappa}{x^{-2}} \qquad (32)$$

$$T_{--}^{(c)}(x^-) = 0 \qquad (33)$$

Note that, since $u' = \lambda^2(\kappa/u^2 - T_{++}(u))^{-1} < 0$ (the flux is above the critical flux), the outgoing flux in region (b) carries negative energy. It can be shown that the time interval during which negative energy emission takes place is less than $O(\lambda^{-1})$, i.e., a "Planckian" interval of time, with Planck-order energy $O(-\lambda\kappa)$ (the total energy emitted by the black hole is always positive [7]).

In region (a) the solution was not modified, so one has $T_{--}^{(a)} = T_{--}^{(i)}$. In region (b) we have

$$T_{--}^{(b)} = T_{--}^{(i)} - T_{--}^{(ii)} \equiv \tilde{T}_{--}^{(b)} \qquad (34)$$

in exact agreement with the formulas obtained by analytically continuing the expressions corresponding to the subcritical regime.

Let us now proceed by considering the case $p > |x_0|$ (which includes macroscopic black holes with $p \gg |x_0|$). The final $\tau \to \infty$ geometry is obtained by taking the limit $x^+ \to \infty$ and $x^- \to -p$ at fixed $x^+ (x^- + p)$ in Eqs. (28), (29). We obtain

$$\Omega^{(b)}(x^+, x^-) = -\lambda^2 x^+(x^- + p) + \kappa \ln(-\lambda^2 x^+(x^- + p)) + \frac{m_f}{\lambda} \tag{35}$$

$$m_f = M(x_2^+) + \lambda\kappa \ln\left(1 - \frac{P_+(x_2^+)}{\lambda^2 p}\right), \qquad x_2^+ \equiv u(-p) \tag{36}$$

This is a static geometry with ADM mass equal to $m_f$. In the whole available space-time, where $-\lambda^2 x^+(x^- + p) > \kappa$, the logarithmic term in Eq. (35) can be neglected and the geometry is essentially the same as the classical black hole geometry. The logarithmic term is only significant close to the line $x^- = -p$, where there is a singularity. However, this lies beyond the boundary at the apparent horizon.

Let us check that energy is conserved. By explicit integration of Eqs. (31) and (32) we now obtain

$$E_{\text{out}}^{(a)} = -\lambda \int_{\infty}^{x_1^-} dx^- \, (x^- + p) T_{--}^{(a)} = -\frac{\kappa p}{x_1^-} - \kappa\lambda \ln\left(1 + \frac{p}{x_1^-}\right)$$

$$E_{\text{out}}^{(b)} = -\lambda \int_{x_1^-}^{-p} dx^- \, (x^- + p) T_{--}^{(b)} = m - M(x_2^+) + \frac{\kappa p}{x_1^-} + \lambda\kappa \ln \frac{1 + p/x_1^-}{1 - P_+(x_2^+)/\lambda^2 p}$$

so that

$$E_{\text{out}}^{(a)} + E_{\text{out}}^{(b)} = m - M(x_2^+) - \lambda\kappa \ln\left(1 - \frac{P_+(x_2^+)}{\lambda^2 p}\right) = m - m_f$$

Thus energy is indeed conserved.

It is clear that for large black holes the total radiated energy is very small. Indeed, the energy that comes out is approximately equal to the energy that came in between times $x_2^+ < x^+ < x_1^+$. But this is a Planckian interval of time, since these points coincide in the classical limit. As an example, we have evaluated $m_f$ for a constant incoming flux $T_{++} = \mathscr{E}/x^{+2}$. We find

$$m_f = (1 - \kappa/\mathscr{E})m, \qquad \text{if} \quad x_1^+/x_0^+ \gg 1$$

$$m_f = m - \kappa\lambda \log m/\lambda, \qquad \text{if} \quad x_1^+/x_0^+ \cong 1$$

so that $m - m_f << m$, since $\mathscr{E} >> \kappa$ for $p >> |x_0^-|$.

To summarize, energy is conserved in both intermediate and supercritical regimes. For small black holes the final state is the vacuum (i.e., they evaporate completely). For macroscopic black holes ($p >> |x_0^-|$) the final state is a static black hole of approximately the same mass as the original configuration. The total emitted energy is a small Planck-order quantity.

## 4. *S*-MATRIX

In this section we will attempt the construction of an *S*-matrix based on the map between ingoing and outgoing modes defined by the reflection condition of the subcritical theory. We start by expanding the matter fields $f(\sigma^+, \sigma^-) = f_+(\sigma^+) + f_-(\sigma^-)$ in terms of Fourier modes[2]

$$\partial_+ f_+(\sigma^+) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \alpha(\omega) e^{i\omega\sigma^+} \qquad \text{(in)} \tag{37}$$

$$\partial_- f_-(\sigma^-) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \beta(\omega) e^{i\omega\sigma^-} \qquad \text{(out)} \tag{38}$$

The canonical commutation relations for $f$ give

$$[\alpha_i(\omega_1), \alpha_j(\omega_2)] = \omega_1 \delta_{ij}\, \delta(\omega_1 + \omega_2) \tag{39}$$

$$[\beta_i(\omega_1), \beta_j(\omega_2)] = \omega_1\, \delta_{ij} \delta(\omega_1 + \omega_2) \tag{40}$$

It is convenient to work in Kruskal coordinates. The Fourier expansions take the form

$$\partial_+ f_+(x^+) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \alpha(\omega)\, (x^+)^{i\omega - 1} \qquad \text{(in)} \tag{41}$$

$$\partial_- f_-(x^-) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \beta(\omega)\, (-x^- - p)^{-i\omega - 1} \qquad \text{(out)} \tag{42}$$

The reflecting boundary condition $dx^+\, \partial_+ f_+(x^+) = dx^-\, \partial_- f_-\, (x^-(x^+))$, where $x^-(x^+)$ is given by Eq. (16), formally relates ingoing and outgoing modes. Now the coordinate $x^-$ is itself an operator since it depends on the operator $P_+(x^+)$ through Eq. (16). Using Eqs. (41), (42), and (16), we obtain

$$\beta(\omega) = \int_0^{\infty} \frac{dx^+}{x^+} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \alpha(\omega')(\kappa - x^+ \bar{P}_+(x^+))^{i\omega} (x^+)^{i(\omega' - \omega)} \tag{43}$$

[2] In what follows we set $\lambda = 1$. For the sake of clarity, the indices $i, j, k = 1, \ldots, N$ will be omitted from some formulas.

where

$$\bar{P}_+(x^+) = \int_{x^+}^{\infty} dx^+ \frac{1}{2} \sum_{i=0}^{N} :\partial_+ f_i \, \partial_+ f_i:, \qquad P_+(x^+) + \bar{P}_+(x^+) = p$$

$$\bar{P}_+(x^+) = -\frac{1}{4\pi^2} \int_{-\infty}^{\infty} \frac{d\omega' \, d\omega'' \, x^{+\,i(\omega'+\omega'')-1}}{i(\omega' + \omega'') - 1} :\alpha_i(\omega')\alpha_i(\omega''):$$

Normal ordering is defined as usual by placing annihilation operators to the right. The "in" vacuum Fock state is annihilated by the $\alpha^i(\omega) = 0$, $\omega > 0$.

Equations (39) and (40) indicate that the transformation (43) must be a canonical transformation and thus the $S$-matrix should be unitary. However, to define fully the operator $\beta(\omega)$ in Eq. (43) [and give a precise mathematical meaning to (43)], it is necessary to expand the binomial. For the low-energy sector we expect $x^+\bar{P}_+$ to have small eigenvalues, and hence we may define this operator $\beta(\omega)$ by expanding in powers of $x^+\bar{P}_+$. As shown in the similar construction in ref. 5, this procedure does not provide a complete description of the out space. The construction gives a nonunitary $S$-matrix representing a small probability of flux coming out of the black hole. This seems consistent with the fact that the analytic continuation of the outgoing energy-momentum tensor of the subcritical regime leads to stable final black hole configurations.

Expanding in powers of $(1/\kappa)x^+\bar{P}_+$, we see that $\beta(\omega)$ in Eq. (43) is of the form

$$\beta(\omega) = \kappa^{i\omega}(\alpha + \kappa^{-1} \, \alpha\alpha\alpha + \ldots + \kappa^{-n} \, \alpha^{2n+1} + \ldots)$$

$$\beta(\omega) = \beta^{(1)}(\omega) + \beta^{(3)}(\omega) + \ldots + \beta^{(2n+1)}(\omega) + \ldots$$

where the $\beta^{(2n+1)}(\omega)$ can be explicitly obtained from Eq. (43) by expanding the binomial. We obtain for the first two terms

$$\beta_j^{(1)}(\omega) = \kappa^{i\omega}\alpha_j(\omega)$$

$$\beta_j^{(3)}(\omega) = \frac{i\omega\kappa^{i\omega}}{4\pi^2\kappa} \int \frac{d\omega' d\omega''}{i(\omega' + \omega'') - 1} \alpha_j(\omega - \omega' - \omega'') :\alpha_k(\omega')\alpha_k(\omega''):$$

Consider, for example, a one-particle "in" state:

$$|1\rangle = \frac{1}{\sqrt{-\omega_2}} \alpha_j(\omega_2)|0\rangle, \qquad \omega_2 < 0$$

For 1–1 and 1–3 processes we obtain the tree-level amplitudes

$$1 \to 1: \quad \left\langle \frac{\beta_i(\omega_1)}{\sqrt{\omega_1}} \, \frac{\alpha_j(\omega_2)}{\sqrt{-\omega_2}} \right\rangle = \kappa^{i\omega_1} \, \delta_{ij} \, \delta(\omega_1 + \omega_2)$$

$$1 \to 3: \quad \langle \beta_i(\omega)\alpha_j(\omega_1)\alpha_k(\omega_2)\alpha_l(\omega_3)\rangle$$
$$= \frac{i}{2\pi^2}\,\omega\omega_1\omega_2\omega_3\kappa^{i\omega-1}\left[\frac{\delta_{ij}\delta_{kl}}{i(\omega+\omega_1)-1} + \text{perm.}\right]$$
$$\times\,\delta(\omega+\omega_1+\omega_2+\omega_3)$$

## 5. BLACK HOLE EVOLUTION IN 3 + 1 DIMENSIONS

Let us consider spherically symmetric configurations of collapsing matter in four-dimensional Einstein gravity coupled to massless matter. We shall first determine the evolution of the apparent horizon, showing, in particular, that there is a low-energy regime in which it is timelike. In this regime, we will examine the implementation of reflecting-type boundary conditions on the apparent horizon (this is motivated by the 1 + 1-dimensional model of Section 3, where such boundary conditions correspond to an analytic transition between subcritical and supercritical regimes). Finally, we will extrapolate the resulting expressions for the outgoing fluxes to the regime where the apparent horizon is spacelike.

We start with metrics of the form (2) and fix the conformal gauge: $g_{ij}(x^0, x^1)\,dx^i\,dx^j = e^{2\rho(U,V)}\,dU\,dV$. To fix the notation, let us consider the static Schwarzschild geometry. The standard connection with Kruskal coordinates $U,V$ is given by

$$2mG(r-2mG)e^{r/2mG} = -V(U+p), \qquad p = 2mG$$
$$U+p = -2mGe^{-u/4mG}, \qquad V = 2mGe^{v/4mG}$$

where $v, u = t \pm r^*$ and $r^* = r + 2mG\log(r-2mG)$. In this case the apparent horizon coincides with the event horizon $U = -p$. For more general, time-dependent configurations, the apparent horizon is determined from the equation

$$\frac{\partial r(U,V)}{\partial V} = 0 \tag{44}$$

Using (44) and the Einstein equation for $g_{VV}$, we obtain

$$\partial_V^2 r - 2\partial_V\rho\partial_V r = -4\pi GrT_{VV}$$

one obtains (see ref. 8 for further details)

$$\frac{dV}{dU}\left[-\frac{kGm^2}{V^2M^2(V)} + \frac{T_{VV}}{\mathcal{T}(V)}\right] = -1, \qquad k = \frac{Ne}{480\pi} \tag{45}$$

where

$$2eG^2 \frac{dM^2(V)}{dV} = V \frac{dP(V)}{dV},$$

$$\frac{dP(V)}{dV} = \frac{T_{VV}}{\mathcal{T}(V)},$$

$$\mathcal{T}(V) \equiv [16\pi e G^3 M^2(V)]^{-1}$$

The parameter $m = M(V_1)$ represents the total ADM mass of the collapsing body, whereas $p = P(V_1)$ is the total infalling Kruskal momentum (we assume that the incident flux starts at $V = V_0$ and stops at $V = V_1$). From Eq. (45) we see that there is a critical value of the incident energy-density flux for which $dV/dU$ changes sign:

$$T^{\text{cr}}_{vv}|_{AH} = N \frac{\pi^2}{30} T_H^4 = \frac{N}{122880\pi^2 G^4 M^4(V)} \tag{46}$$

For lower $T_{VV}$ the apparent horizon is timelike; for greater $T_{VV}$, it is spacelike. Note that a spacelike apparent horizon necessarily involves a black hole geometry, since it implies that the curve $r(U, V) = 0$ is spacelike. For $V$ near $V_1$, one has $M(V) \cong m$, and Eq. (46) reduces to

$$V[U + P(V)] \cong -kG \tag{47}$$

Next, we determine the boundary conditions on a timelike apparent horizon. In the classical theory, a reflection on a boundary $V(U)$ is given by the condition

$$T_{UU} = T_{VV} \left( \frac{dV}{dU} \right)^2$$

In the quantum theory, we need to relate $T_{UU} = {}_{\text{in}}\langle 0|\hat{T}_{UU}|0\rangle_{\text{in}}$ and $T_{VV} = {}_{\text{in}}\langle \alpha|\hat{T}_{VV}|\alpha\rangle_{\text{in}}$, where ${}_{\text{out}}\langle 0|\hat{T}_{UU}|0\rangle_{\text{out}} = 0$ and ${}_{\text{in}}\langle 0|\hat{T}_{VV}|0\rangle_{\text{in}} = 0$, and $|\alpha\rangle_{\text{in}}$ is the quantum state representing collapsing matter. Any two composite operators $\hat{T}_{UU}$ and $\hat{T}'_{UU}$ differing only in normal-ordering substraction will be related by $\hat{T}'_{UU} = \hat{T}_{UU} - \hat{t}_{UU}$, where for free fields $\hat{t}_{UU}$ will be a c-number. Thus we expect for the quantum theory a relation of the form

$$T^{\text{R}}_{UU} = t_{UU} + \left( \frac{dV}{dU} \right)^2 (T_{VV} - t_{VV}) \tag{48}$$

In the regions where $T_{VV} = 0$ the outgoing flux will be purely due to radiation,

$$T^{\rm H}_{UU} = t_{UU} - \left(\frac{dV}{dU}\right)^2 t_{VV} \tag{49}$$

The calculation here will not depend on the explicit form of $t_{UU}$, $t_{VV}$. The radiated energies in the subcritical regime can be calculated by integrating (48) and (49) (cf. Fig. 1, with $x^-, x^+ \to U, V$)

$$E^{(\rm i)}_{\rm out} = 4\pi \int_{-\infty}^{u_0} du\, r^2\, T_{uu} = -\frac{\pi}{mG}\int_{-\infty}^{U_0} dU\,(U+p)r^2\, T^{\rm H}_{UU} \tag{50}$$

$$E^{(\rm ii)}_{\rm out} = -\frac{\pi}{mG}\int_{U_0}^{U_1} dU\,(U+p)r^2\, T^{\rm R}_{UU}$$

Suppose that between $(V_0, V_1)$ $T_{vv}$ is increased above $T^{\rm cr}_{vv}$. The apparent horizon for those $V$ will be spacelike and, as a result, part of region (i) will overlap with region (ii) (cf. Fig. 2). We continue the subcritical expressions just as in the $(1 + 1)$-dimensional model of Section 3. The energy in region (a) is obtained by integrating the usual Hawking flux $T^{\rm H}_{UU}$, whereas the energy radiated between $U_1$ and $U_0$ is given by

$$E^{(\rm b)}_{\rm out} = -\frac{\pi}{mG}\int_{U_1}^{U_0} dU\,(U+p)r^2\, T^{(\rm b)}_{UU} \tag{51}$$

where

$$T^{(\rm b)}_{UU} = T^{\rm H}_{UU} - T^{\rm R}_{UU} = -T_{VV}\left(\frac{dV}{dU}\right)^2 \tag{52}$$

We note that the outgoing energy-density flux between $U_1$ and $U_0$ is negative. It will soon be clear that the amount of negative energy radiated in region (b) is a tiny Planck-scale quantity (e.g., for a solar-mass black hole, $E^{(\rm b)}_{\rm out} = -10^{-114} m_{\rm pl}$). In quantum theory energy-density is not positive definite; in the present case it is the tail of the outgoing wave that carries off this bit of negative energy.

Let us now restrict our attention to macroscopic black holes ($m >> m_{\rm pl}$). The total energy radiated in region (a) can be calculated by integrating the Hawking radiation flux near the horizon,

$$T^{\rm H}_{uu} \sim \frac{1}{(Gmr)^2} \qquad \text{or} \qquad T^{\rm H}_{UU} \sim \frac{1}{r^2(U+p)^2}$$

We find

$$E_{\text{out}}^{(\text{a})} \cong \frac{1}{mG}, \qquad E_{\text{out}}^{(\text{b})} \cong -\frac{1}{m^3 G^2}$$

so that

$$E_{\text{out}}^{\text{total}} = E_{\text{out}}^{(a)} + E_{\text{out}}^{(b)} \cong (mG)^{-1} = \left(\frac{m_{\text{pl}}}{m}\right) m_{\text{pl}}$$

As in the (1 + 1)-dimensional model, the total radiated energy is a small Planck-scale quantity. The black hole retains most of its original mass.

## 6. CONCLUSIONS

Here we have explored the theory that results from analytically continuing the subcritical regime above the threshold of black hole formation. In the corresponding semiclassical theory, quantum effects appear in various ways, but the net result is that only small alterations over a classical picture appear. This may be interpreted as an indication of an ultraviolet softening in quantum gravity, since the quanta that are suppressed are those which have trans-Planckian frequencies at the moment they arise from the horizon. The apparent horizon acts as a natural ultraviolet cutoff. A boundary at the apparent horizon amounts to excluding the region of space-time where the contours $r = \text{const.}$ are spacelike. (It should be stressed that no "wall" prevents an infalling macroscopic object from entering into the black hole. As a macroscopic object is falling in, the apparent horizon expands and the object always remains inside the black hole. It is only the small emitted radiation that is effectively described as if there was a boundary at the apparent horizon.)

The stability of the final geometry can be understood in different ways. It is known that in order to have zero fluxes at infinity [in the present case, in region (b)], the gravitational field must be greatly modified near the line $U = -p$—the Boulware vacuum choice, defined in terms of the Schwarzschild Killing vector [18]. As seen in Section 3, this is exactly what is happening: in the allowed space-time region, the final geometry (35) is essentially the same as the classical black hole geometry; only at distances exponentially close to the event horizon is the geometry significantly modified, but this lies beyond the boundary. Here the geometry has settled down to this situation dynamically having started from the Unruh vacuum.

As far as the information problem is concerned, one may distinguish three plausible scenarios: (1) the Hawking model, where pure states evolve into mixed states, and there is no unitary evolution; (2) stabilized black holes, where Hawking particles corresponding to transplanckian frequencies are suppressed by quantum gravity effects; (3) the 't Hooft picture, where black holes evaporate completely, and quantum gravity effects on transplanckian modes are such that the unitarity of the evolution is preserved.

The present results may be regarded as an implementation of the second scenario, which thus seems to be supported by the analytic properties of an $S$-matrix. There is no loss of quantum coherence, since all quantum mechanical information remains inside the final stable black hole geometry. The effective equivalence between analytic continuation and a theory with a boundary at the apparent horizon may be interpreted as another manifestation of the membrane paradigm [19].

It would be interesting to investigate possible implications for the D-brane description of near-extremal black holes [20]. An extension of the present calculation for charged black holes seems necessary before a comparison with amplitudes obtained by using D-brane methods can be made.

## ACKNOWLEDGMENTS

I would like to thank the organizers of the conference Quantum Gravity in the Southern Cone II, Bariloche, for their kind hospitality. This work is partly supported by the European Commission TMR program grant ERBF-MBI-CT96-0982.

## REFERENCES

[1] S. W. Hawking, *Commun. Math. Phys.* **43** (1975) 199.
[2] G. 't'Hooft, *Nucl. Phys. B* **256** (1985) 727; **335** (1990) 138.
[3] G. 't Hooft, *Int. J. Mod. Phys. A* **11** (1996) 4623.
[4] D. Amati, M. Ciafaloni, and G. Veneziano, *Int. J. Mod. Phys. A* **3** (1988) 1615; *Nucl. Phys. B* **403** (1993) 707.
[5] H. Verlinde and E. Verlinde, *Nucl. Phys. B* **406** (1993) 43; K. Schoutens, H. Verlinde, and E. Verlinde, *Phys. Rev. D* **48** (1993) 2670.
[6] C. Stephens, G.'t Hooft, and B. Whiting, *Class. Quant. Grav.* **11** (1994) 621.
[7] D. Amati and J. G. Russo, *Phys. Rev. D* **56** (1997) 974.
[8] J. G. Russo, *Phys. Rev. D* **55** (1997) 871.
[9] S. W. Hawking, *Phys. Rev. D* **14** (1976) 2460.
[10] T. Jacobson, *Phys. Rev. D* **44** (1991) 1731.
[11] L. Susskind, L. Thorlacius, and J. Uglum, *Phys. Rev. D* **48** (1993) 3743.
[12] Y. Kiem, H. Verlinde, and E. Verlinde, *Phys. Rev. D* **52** (1995) 7053.
[13] G. 't Hooft, gr-qc/9706058.
[14] G. 't Hooft, In *Salamfest*, A. Ali *et al.*, ed. [gr-qc/9310026 ]; L. Susskind, *J. Math. Phys.* **36** (1995) 6377.
[15] J. G. Russo, *Phys. Lett. B* **359** (1995) 69.

[16] C. Callan, S. Giddings, J. Harvey, and A. Strominger, *Phys. Rev. D* **45** (1992) 1005.
[17] J. G. Russo, L. Susskind, and L. Thorlacius, *Phys. Rev. D* **46** (1992) 3444; **47** (1993) 533.
[18] D. G. Boulware, *Phys. Rev. D* **11** (1975) 1404.
[19] K. Thorne, R. Price, and D. MacDonald, *Black Holes: The Membrane Paradigm*, Yale University Press, New Haven, Connecticut (1986).
[20] J. Maldacena, *Nucl. Phys. Proc. Suppl.* **61A** (1998) 111.